$$\phi^{(m)}(x)=b_m.m!+b_{m+1}.\frac{(m+1)!}{1!}x+\ \ldots\ldots\ +b_s.\frac{s!}{(s-m)!}x^{s-m}.$$

If we add together the equations (3), we evidently obtain as the sum of the terms in the main diagonal, from $b_1 1!$ to $b_s.s!.\frac{x_k^{s-1}}{(s-1)!}$, the polynomial $\phi'(x_k)$; as the sum of the terms in the next lower diagonal $\phi''(x_k)$, etc. We therefore have

$$e^{x_k}(1!b_1+2!b_2+\ldots+s!b_s)=\phi'(x_k)+\phi''(x_k)+\ldots+\phi^{(s)}(x_k)+\sum_{m=1}^{s}b_m x_k^m R_{km} \quad (4)$$

in which $R_{km}=1+\frac{x_k}{m+1}+\frac{x_k^2}{(m+1)(m+2)}+\ldots\ldots$

Suppose now that $\phi(x)$, which is perfectly arbitrary, be chosen as below so that

$$\phi'(x_k)=0,\quad \phi''(x_k)=0,\ \ldots\ldots,\ \phi^{(p-1)}(x_k)=0,$$

for every x_k, $p<s$. By returning to the arrangement of (3) and leaving out the terms due to $\phi'(x_k)$, ..., $\phi^{(p-1)}(x_k)$, we could then rewrite (4) in the form

$$\begin{aligned}e^{x_k}(1!b_1+2!b_2+\ldots\ldots+s!b_s)=&\sum_{m=1}^{s}b_m x_k^m R_{km}\\ &+b_p.p!\\ &+b_{p+1}.(p+1)!(1+\frac{x_k}{1!})\\ &+b_{p-2}.(p+2)!(1+\frac{x_k}{1!}+\frac{x_k^2}{2!})+\ldots\ldots\\ &+b_s.s!(1+\frac{x_k}{1!}+\frac{x_k}{2!}+\ldots+\frac{x_k^{s-p}}{(s-p)!}).\end{aligned} \quad (5).$$

A choice of $\phi(x)$ that satisfies the conditions just required is

$$\phi(x)=\frac{x^{p-1}}{(p-1)!}(a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n)^p\equiv\frac{x^{p-1}(f(x))^p}{(p-1)!}$$

of which every x_k is a p-tuple root, by (2). Here p is still perfectly arbitrary, but $s=np+p-1$, the degree of $\phi(x)$. Expanding $\phi(x)$, we find on account of the factor x^{p-1}

$$b_0=0,\ b_1=0,\ \ldots\ldots,\ b_{p-2}=0,$$

$$b_{p-1}=\frac{a_0{}^p}{(p-1)!},\quad b_p=\frac{I_p}{(p-1)!},\quad \ldots\ldots,\quad b_s=\frac{I_s}{(p-1)!},$$

where I_p, ..., I_s are all integers.

Now the coefficient of e^{x_k} in (5) evidently becomes

$$N_p=a_0{}^p+\frac{I_p}{(p-1)!}\cdot p!+\frac{I_{p+1}}{(p-1)!}\cdot(p+1)!+\ldots\ldots+\frac{I_s\,s!}{(p-1)!}.$$

If the arbitrary p is taken as a prime number greater than a_0, this expression is the sum of $a_0{}^p$, which cannot contain p as a factor, plus a number of other integers each of which does contain the factor p. N_p is therefore *not zero and not divisible by p.*

Further, since $(p+k)!\div[(p-1)!\ k!]$ is an integer divisible by p, it follows that all of the coefficients of the last block of terms in (5) contain p as a factor. On adding the columns of (5) we have:

$$N_p e^{x_k}=p[P_0+P_1x_k+P_2x_k{}^2+\ldots\ldots+P_{s-p}(x_k)^{s-p}]+\sum_{m=1}^{s}b_m x_k R_{km}, \qquad (6)$$

where P_0, P_1, ..., P_{s-k} are integers.

Before completing our argument we need only to show that by choosing as p a prime number sufficiently large, the last term of (6) can be made as small as we please. If a is a number greater than unity and greater than any of the n roots x_k of $f(x)$,

$$|R_{km}|=\left|1+\frac{x_k}{m+1}+\frac{x_k{}^2}{(m+1)(m+2)}+\ldots\ldots\right|<\left|1+\frac{a}{1!}+\frac{a^2}{2!}+\ldots\ldots\right|.$$

$$\therefore\ |R_{km}|<e^a.$$

Now since the coefficients b_m in (6) are the coefficients of $\phi(x)$ and since each coefficient of $\phi(x)$ is numerically less than or equal to the corresponding coefficient of

$$\frac{x^{p-1}}{(p-1)!}(|a_0|+|a_1|x+|a_2|x^2+\ldots\ldots+|a_n|x^n)^p,$$

we have the inequality, Q denoting a constant,

$$\left|\sum_{m=1}^{s}b_m x_k{}^m R_{km}\right|<e^a\cdot\frac{a^{p-1}}{(p-1)!}(|a_0|+|a_1|a+\ldots\ldots+|a_n|a)^p<\frac{(Q)^p}{(p-1)!}.$$

The last expression, designated Σ_p, is the pth term of the series for Qe^Q and therefore approaches zero as p is increased indefinitely.

We now choose the arbitrary prime number $p>1$ so that it shall be larger that a_0, larger than C, and also so that $\Sigma_p<1/n$. *The number N_p is the required multiplier N.*

For if we multiply N_p into (1) in follows directly from equation (6) that

$$N_p(C+e^{x_1}+e^{x_2}+\ldots\ldots+e^{x_n})=N_pC+p(P_0+P_1S_1+P_2S_2+\ldots\ldots+P_{s-p}S_{s-p})+r_1+r_2+\ldots\ldots+r_n \qquad (7)$$

where $r_k=\sum_{m=1}^{s} b_m(x_k)^m R_{km}<1/n$, $S_i=x^i{}_1+x^i{}_2+\ldots\ldots+x^i{}_n$.

But from Newton's formulas*

$$S_1+a_1=0, \quad S_2+a_1S_1+2a_2=0, \ \ldots\ldots$$

it follows that $S_1, S_2, \ldots\ldots, S_{s-p}$ are whole numbers. Hence the second term of the right-hand member of (7) is an integer divisible by p. On the contrary, N_p and C are not divisible by p. The sum of these terms therefore is a whole number greater than $+1$ or less than -1; and since the sum $r_1+r_2+\ldots\ldots+r_n$ is less than unity the right-hand member of (7) cannot be zero. Hence the left-hand member of (7) is not zero and hence (1) cannot be zero.

§3. The proof that e is a transcendental number can be effected by almost precisely the same argument as that given above. It is required to show that the algebraic equation with integral coefficients

$$c+c_1e+c_2e^2+\ldots\ldots+c_ne^n=0 \qquad (1')$$

is impossible. Evidently no generality is lost by assuming $c\neq 0$ and $c_n\neq 0$. Let

$$f(x)=(x-1)(x-2)\ldots\ldots(x-n)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n. \qquad (2')$$

The argument now is exactly like that of §2 from equation (2) to the sentence introducing equation (7). At this point we observe that since all the roots of $f(x)$ are integers, (6) may be written

$$N_pe^{x_k}=pW_k+r_k,$$

where W_k is a whole number and r_k is less than $1/n$. We therefore have

$$N_p(c+c_1e+\ldots\ldots+c_ne^n)=c.N_p+p(W_1+W_2+\ldots\ldots+W_n)+r_1+r_2+\ldots\ldots+r_n. \qquad (7')$$

In the right-hand member, the first term is not divisible by p, the second term is divisible by p and the third term is numerically less than unity. From this it follows as before that the left-hand member of (7') cannot be zero and hence that (1') is impossible. Therefore e cannot satisfy an algebraic equation.

*Cf. Burnside and Panton, *Theory of Equations*, Chapter VIII, or any book on higher algebra.

A BALANCE FOR THE SOLUTION OF ALGEBRAIC EQUATIONS.

By R. P. BAKER, Chicago.

The ordinary balance with a riding weight and divided beam solves the equation $ax+b=0$. Here a represents the weight of the rider, x its distance from the zero of the scale positive or negative, and b the weight in one of the pans, positive or negative, according to the pair.

Consider the quadratic $F\equiv ax^2+bx+c=0$, with a, b, c, real, and a positive. Take three balance beams parallel and in same horizontal plane; their fulcrums in a line perpendicular to the beams. Let a rider of weight a be placed on the first beam, and a fixed weight b at unit distance right or left according to sign of b (or unit weight at distance b). If x is the distance of the rider from zero of the scale the moment turning the beam is $ax+b$. By some mechanism, approximately and conveniently by a long thread attached at distance unity on this beam and passing vertically both up and down (to transmit forces of either sign) over pulleys sufficiently distant to preserve approximate parallelism, a force proportional to $ax+b$ is transmitted to the second beam at a distance x from its fulcrum. The second beam also has a weight c at unit distance of appropriate sign (or unit weight at distance c). The moment on the second beam is now ax^2+bx+c. This may have either sign, but the sign of this does not tell us which way to vary x to reach a zero value, or whether one can be reached. The third beam has weight $2a$ at distance x, and b at unit distance, and has turning moment $2ax+b\equiv\partial F/\partial x$. If this is positive x must be made to move in the negative direction, or vice versa, to diminish the value of F. The third beam may be stopped and its bearing on the stops is the indicator as in the ordinary balance.

If the equation has imaginary roots the third or indicator beam reverses its moment without the second beam coming to an equilibrium. The three riders on the three beams can be moved so as to remain in a straight line perpendicular to the beams by an appropriate frame and vertical forks containing the axes of small rollers carrying the riders. The machine may be extended in principle to an equation of degree n, using $2n-1$ beams, and two indicators for the signs of x and $F(x)$.

The machine may be made automatic by electrical attachments. In the case of the quadratic, when the sign of $F'(x)$ is positive, and $F(x)$ is positive, a connection through the stops may be used to start a rotation which moves the x-frame in the negative direction. For each change of sign the connection would reverse the current so that the sign of $-F(x)F'(x)$ gives the direction of movement of the x-frame. In case of imaginary roots the machine after some oscillation would come to rest in the minimum position.

GROUPS OF ELEMENTARY TRIGONOMETRY.

By G. A. MILLER.

In a preceding number of this journal we considered a substitution group of degree 6 and order 12, which is very useful in deriving formulas from each other.* In the present note we shall consider some groups of operations, which are associated with the triangle and the trigonometric functions. Nearly all of these groups are too elementary to exhibit many group properties, yet it is hoped that their formal statement will exhibit some important points in a clear light.

We begin with the operations upon the angle α which transform $\sin\alpha$ into $\pm\sin\alpha$. By reflecting the point representing α on the y-axis† we obtain a point representing an angle whose sine is equal to $\sin\alpha$; by reflecting the same point on the x-axis there results an angle whose sine is equal to $-\sin\alpha$. These two reflections, in succession, are equivalent to a rotation through the angle π, which transforms α into another angle whose sine is equal to $-\sin\alpha$. It is well known that the three given operations together with the identity constitute the four-group, or the axial group. Hence this axial group transforms the angle α so that $\sin\alpha$ goes into $\pm\sin\alpha$. It is clear that the same operations transform each of the other five trigonometric functions into $\pm$ the same named function.

By reflecting on the bisector of the first quadrant, every angle is changed into its complement. Hence the direct functions of α will be transformed into the co-functions by this operation. The group generated by this operation and the preceding axial group is the octic group, or the group of movements of the square. Hence this octic group transforms α in such a way that each function of α will go into $\pm$ itself or into $\pm$ its co-function. Since subtracting α from any angle β is equivalent to reflecting α on the diameter through $\beta/2$, it follows that the given operations result from taking the complement and supplement of the angle α, these operations being performed in any order. Hence by taking the complement or the supplement of an angle any number of times and in any order, we transform any of its six functions either into $\pm$ itself or into $\pm$ its co-function.

Now angles β_1, β_2 can be so selected that the two operations of subtracting α from them generate a dihedral rotation group of any even order, since the order of this group is $2\pi/d$, where d is the highest common factor of β_1, β_2, and 2π.‡ Theoretically, these groups are just as interesting as the group formed by taking the complement and supplement of the angle α, but they do not have such important applications in the solution of the triangle and hence need not be considered here.

In the preceding cases the operations of the group were performed upon the angle α. Instead of this, the operations may be performed upon functions of α. Perhaps the most interesting case results when the operations of inversion

*Vol. 5 (1898), p. 102.

†The angles are supposed to be represented by points on the unit circle.

‡Cf. Groups of Subtraction, MONTHLY, NOV., 1904.

and subtracting from unity are performed upon $\sin^2 a$. The six operations of the group generated by the two given operations* transform $\sin^2 a$ into the following functions of $\sin^2 a$:

$$\sin^2 a, \quad 1-\sin^2 a, \quad \frac{1}{\sin^2 a}, \quad \frac{\sin^2 a-1}{\sin^2 a}, \quad \frac{\sin^2 a}{\sin^2 a-1}, \quad \frac{-1}{\sin^2 a-1}.$$

These are respectively,

$$\sin^2 a, \quad \cos^2 a, \quad \csc^2 a, \quad -\cot^2 a, \quad -\tan^2 a, \quad \sec^2 a.$$

Each of these functions is transformed into all the others by the operations of this group.

A more elementary example presents itself if it is observed that the four-group is the group of division and changing sign. In particular, it is the group of inversion and changing sign. Hence the three sets of four functions ($\pm\sin a$, $\pm\csc a$; $\pm\tan a$, $\pm\cot a$; $\pm\sec a$, $\pm\csc a$) are transformed among themselves by the operations of this group.

If the three sides of a triangle are regarded as directed lines there are eight triangles having the same vertices. As the operations of changing the direction of the sides are independent operations of period two, it follows that each of these triangles may be transformed into all of them by the operations of the group of order 8 which contains seven operations of period two. Similarly, we obtain 64 trihedral angles from a given one by giving direction to the edges and the face angles. The corresponding 64 spherical triangles can be obtained from any one of them by the 64 operations of the group containing 63 operations of period two. These triangles have been studied in details by Study, *Sächsischen Abhandlungen*, Vol. 20, pp. 87-231. A very brief account of them is found in Jones' *Trigonometry*, 1896, p. 168.

The main interest in employing group theory in the study of trigonometry is due to the fact these same groups have extensive application in other fields of mathematics. For instance, by means of the given operations of subtracting from unity and inverting we associate, in general, six expressions with every expression. The six expressions associated with $\sin^2 A/2=\frac{(s-b)(s-c)}{bc}$ are

$$\frac{(s-b)(s-c)}{bc}, \quad \frac{s(s-a)}{bc}, \quad \frac{bc}{(s-b)(s-c)}, \quad \frac{s(a-s)}{(s-b)(s-c)}, \quad \frac{(s-b)(s-c)}{s(a-s)}, \quad \frac{bc}{s(s-a)}.$$

These are respectively,

$$\sin^2\frac{A}{2}, \quad \cos^2\frac{A}{2}, \quad \csc^2\frac{A}{2}, \quad -\cot^2\frac{A}{2}, \quad -\tan^2\frac{A}{2}, \quad \sec^2\frac{A}{2}.$$

*Subtracting from x_1 and dividing x_1^2 gives rise to this dihedral rotation group for every value of x_1. In the present case $x_1=1$.

No new expressions are obtained by subtracting these from unito or by inverting them. The study of equality among these functions becomes a special case when the group generated by the operations $x_1 - n, x_1^2/n$ transforms a point into less than six distinct points. This question has been completely solved for the general dihedral rotation group.

THE EXPRESSION OF THE AREAS OF POLYGONS IN DETERMINANT FORM.

By R. P. BAKER.

The area of the triangle the rectangular coördinates of whose vertices are (x_1, y_1); (x_2, y_2); (x_3, y_3), is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For the area of a quadrilateral whose vertices are 1, 2, 3, 4, diagonals (13) and (24) and such that circuits (123), (134) have the area on the left, we have

$$\frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & 0 \\ x_2, & y_2, & 1, & 1 \\ x_3, & y_3, & 1, & 0 \\ x_4, & y_4, & 1, & 1 \end{vmatrix} \equiv \frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1, & -k \\ x_2, & y_2, & 1, & 1-k \\ x_3, & y_3, & 1, & -k \\ x_4, & y_4, & 1, & 1-k \end{vmatrix}.$$

The case of the pentagon or polygon of more sides than 5 is different.

Suppose that the area P of the pentagon can be expressed by

$$\lambda\begin{vmatrix} x_1, & y_1, & 1, & a_1, & b_1 \\ x_2, & y_2, & 1, & a_2, & b_2 \\ x_3, & y_3, & 1, & a_3, & b_3 \\ x_4, & y_4, & 1, & a_3, & b_4 \\ x_5, & y_5, & 1, & a_5, & b_5 \end{vmatrix} = \lambda\begin{vmatrix} x_1, & y_1, & 1, & 0, & 0 \\ x_2, & y_2, & 1, & a'_2, & b'_2 \\ x_3, & y_3, & 1, & a'_3, & b'_3 \\ x_4, & y_4, & 1, & a'_4, & b'_4 \\ x_5, & y_5, & 1, & a'_5, & b'_5 \end{vmatrix}$$

the latter being obtained from the former by subtracting multiples of columns.

Expanding by Laplace's method in minors of the first three columns we get the area as a sum of multiples of triangular areas all having the point 1 as vertex. The multipliers must obviously be equal. Hence all the determinants $\begin{vmatrix} a'_i & b'_i \\ a'_j & b'_j \end{vmatrix}$ must be equal. This is impossible, for

$$\tfrac{1}{2}\begin{vmatrix} a_2, & b_2, & a_2, & b_2 \\ a_2, & b_3, & a_3, & b_3 \\ a_4, & b_4, & a_4, & b_4 \\ a_5, & b_5, & a_5, & b_5 \end{vmatrix} \equiv (23)(45)-(24)(35)+(25)(34) \equiv 0,$$

which cannot be satisfied by

$$(23)=(45)=(24)=(35)=(25)=(34).$$

The general case fails in consequence of a similar identical relation among the determinants of a matrix. This relation can be expressed as the expansion of a determinant of $2(n-3)$ rows which can be symbolized by $\begin{matrix} A \mid C \\ B \mid C \end{matrix}$, where A, B, C denote, respectively,

$$\begin{vmatrix} 0, & 0, & \ldots, & a_{n-2} \\ 0, & 0, & \ldots, & b_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & l_{n-2} \end{vmatrix}, \quad \begin{vmatrix} a_2, & a_3, & a_4, & \ldots, & a_{n-2} \\ b_2, & b_3, & b_4, & \ldots, & b_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_2, & l_3, & l_4, & \ldots, & l_{n-2} \end{vmatrix}, \quad \begin{vmatrix} a_{n-1}, & a_n, & a_2, & a_3, & \ldots, & a_{n-4} \\ b_{n-1}, & b_n, & b_2, & b_3, & \ldots, & b_{n-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ l_{n-1}, & l_n, & l_2, & l_3, & \ldots, & l_{n-4} \end{vmatrix}.$$

When this is expanded according to Laplace's method in determinants of $(n-3)$ rows, we get

$$\begin{aligned} &(2, 3, \ldots\ldots, n-3, n-2)(n-1, n, 2, \ldots\ldots, n-4) \\ -&(2, 3, \ldots\ldots, n-3, n-1)(n-2, n, 2, \ldots\ldots, n-4) \\ +&(2, 3, \ldots\ldots\ldots, n-3, n)(n-3, n, 2, \ldots\ldots, n-4)=0, \end{aligned}$$

which cannot be satisfied if these determinants are all equal.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problem 206 was also solved by J. E. Sanders; No. 207 was also solved by L. E. Newcomb.

208. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve (1) ..... $x^4+y^4=14x^2y^2$; (2) ..... $x+y=m$.

I. Solution by EDWIN L. RICH, Student at Lehigh University.

Equation (1) may be written

$$(x^2-y^2-xy\sqrt{12})(x^2-y^2+xy\sqrt{12})=0 \ldots\ldots (3).$$

Solving each factor,

$$x=(\sqrt{3}\pm 2)y \quad \ldots\ldots (4),$$
$$x=(-\sqrt{3}\pm 2)y \quad \ldots\ldots (5).$$

From the simultaneous equations (2), (4) and (2), (5),

$$x=\frac{m}{6}(3\pm\sqrt{3}),\quad y=\frac{m}{6}(3\mp\sqrt{3}),$$

$$x=\frac{m}{2}(1\pm\sqrt{3}),\quad y=\frac{m}{2}(1\mp\sqrt{3}).$$

II. Solution by S. A. COREY, Hiteman, Iowa.

Adding $2x^2y^2$ to each member of (1),

$$x^4+2x^2y^2+y^4=16x^2y^2 \quad \ldots\ldots (3),$$

whence $x^2+y^2=\pm 4xy \quad \ldots\ldots (4).$

Squaring (2), we have

$$x^2+2xy+y^2=m^2 \quad \ldots\ldots (5);$$

substituting from (4), $6xy=m^2$, or $-2xy=m^2$,

whence, $x=\frac{m^2}{6y}$ or $-\frac{m^2}{2y}$.

By substituting these values of x in (2), we find without difficulty,

$$x=m(\tfrac{1}{2}\pm\sqrt{\tfrac{1}{12}}),\text{ or } m(\tfrac{1}{2}\pm\sqrt{3}/4);$$
$$y=m(\tfrac{1}{2}\mp\sqrt{\tfrac{1}{12}}),\text{ or } m(\tfrac{1}{2}\mp\sqrt{3}/4).$$

Also solved by R. L. Borger, G.W. Greenwood, E. L. Sherwood, L. E. Newcomb, J. F. Lawrence, S. E. Harwood, Elmer Schuyler. G. B. M. Zerr, and J. Scheffer.

209. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prove that $(a^4+b^4+c^4+d^4)>4abcd$.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, O.

$(a^2+b^2)>2ab$ and $(c^2+d^2)>2cd$ from which, by multiplication, $(a^2c^2+b^2c^2+a^2d^2+b^2d^2)>4abcd$ ..... (1).

Again $(a^4+c^4)>2a^2c^2$; $(b^4+c^4)>2b^2c^2$; $(a^4+d^4)>2a^2d^2$, and $(b^4+d^4)>2b^2d^2$.

By addition, $(a^4+b^4+c^4+d^4)>(a^2c^2+b^2c^2+a^2d^2+b^2d^2)$ and this in connection with (1) gives $(a^4+b^4+c^4+d^4)>4abcd$.

Also solved by G.W. Greenwood, R. L. Borger, E. L. Sherwood, L. E. Newcomb, Elmer Schuyler, J. H. Meyer, A. J. Haun, C. A. Laisant, J. Scheffer.

*** Dr. G. B. M. Zerr and Mr. J. F. Lawrence prove in general that $a_1+a_2+\ldots\ldots+a_n>n\sqrt{(a_1a_2\ldots\ldots a_n)}$. Dr. Zerr also proved that $(a_1^m+\ldots\ldots+a_n^m)/n>[(a_1+\ldots\ldots+a_n)/n]^m$.

210. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The sum of five quantities and the sum of their cubes are both zero. Show that the sum of their fifth powers is a factor of the sum of any odd powers of the quantities.

Solution by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Denote the quantities by $\alpha, \beta, \gamma, \delta, \epsilon$, and let the equation of which they are the roots be

$$x^5+ax^4+bx^3+cx^2+dx+e=0.$$

Then
$$\Sigma\alpha=-a=0.$$
$$\Sigma\alpha^3=(\Sigma\alpha^2-\Sigma\alpha\beta)\Sigma\alpha-\Sigma\alpha\beta\gamma)=c=0.$$

Substituting the roots in turn for x, and adding, we get,

$$\Sigma\alpha^5+5e=0.$$

Multiply the equation by x^2, make the same substitutions, and we get in a similar manner,

$$\Sigma\alpha^7+b\Sigma\alpha^5+5e=0,$$
$$i.\ e.,\ \Sigma\alpha^7+(b-1)\Sigma\alpha^5=0.$$

Hence $\Sigma\alpha^5$ is a factor of $\Sigma\alpha^7$ and the process may be repeated indefinitely.

Also solved by Elmer Schuyler, G. B. M. Zerr, J. Scheffer.

GEOMETRY.

228 Proposed by O.E. GLENN, A.M., Fellow in Mathematics, University of Pennsylvania, Philadelphia, Pa

Given a point O without a circle S; two arbitrary lines through O cut S in the points A, A', and B, B', respectively. Prove, by pure geometry, that the four circles through OAR, OBR, $OA'R'$, $OB'R'$, respectively, intersect in points collinear with O; R and R' being points upon S arbitrarily chosen.

Solution by T. L. CROYES, Paris, France.

Let us take the inverse of the system with regard to O, and let the inverses of the five circles S, OAR, OBR, $OA'R'$, $OB'R'$, be a circle s, and the four right lines ar, br, $a'r'$, $b'r'$ (a, a', b, b', r, r' being the inverses of A, A', B, B', R, R').

By Pascal's theorem, the points of intersection $(ar, b'r')(br, a'r')$ are collinear with the point O.

237. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plain or gauche. Double the length of CB and DE by extending from B and E to G and H, re-

spectively. Draw $B'D$ parallel to and of the same currency as BC. Connect G and H. Then prove that $2(AB^2+BC^2+CD^2+DE^2+EA^2)=3CD^2+4(DE.BC.\cos EDB'+EA.AB.\cos EAB)+GH^2$.

Solution by the PROPOSER.

Let a, b, c, and d be the vector sides corresponding with DE, EA, AB, and BC, respectively. Then will $-(a+b+c+d)$ represent the side CD, and $(a-b-c+d)$, the line GH. Squaring each of these six vectors to get their squared tensors, and adding as follows:

$$2[T^2a+T^2b+T^2c+T^2d+T^2.(-a-b-c-d)]$$
$$-3T^2.(-a-b-c-d)-T^2.(a-b-c+d)$$

is found to be $4(Ta.Td.\cos ad+Tb.Tc.\cos bc)$ as required by the problem.*

238. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Construct a trapezoid having given the sum of the parallel sides, the sum of the diagonals, and the angle formed by the diagonals.

I. Solution by J. H. MEYERS, S. J., Professor of Mathematics in Spring Hill College, Mobile, Ala.

Construct the triangle ABC, whose base AB=sum of parallel sides, angle C=angle between diagonals and where $AC+CB$=sum of diagonals. [See Casey's *Sequel to Euclid*, Book III, Prop. 29.]†

Take the point E on AB, such that $CE=CB$; from E draw EF parallel and equal to AC meeting CB in O; join B and F. $CFBE$ is the required trapezoid. In proof, $AE=CF$, hence $EB+CF$=given sum of parallel sides. Since $EF=AC$, $EF+CB$=given sum of parallel sides. And finally, angle $EOB=ACB$.

Analogously solved by J. Scheffer.

II. Remark by G. W. GREENWOOD, M. A., Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

Consider any trapezoid $ABCD$ satisfying the given conditions, where AB, CD are the parallel sides. Draw any parallel to AC meeting AB internally in A', say, and hence meeting CD internally in C'. The trapezoid $A'BC'D$ can easily be seen to fulfill the required conditions. Therefore the solution is not unique.

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*In the above solution the lines CB and DE are the ones produced. Prof. G. W. Greenwood has demonstrated that the corresponding theorem, where the lines produced are BC and ED, does not hold true. Ed. E.

†Describe on the same side of AB as a chord the arcs AXB, AYB such that angle AXB=given angle α for every point X, and $AYB=\frac{1}{2}\alpha$. Determine the point P on arc AYB such that AP=sum of diagonals of trapezoid (*i. e.*, sum of sides of the required triangle). Let AP meet arc AXB in C. Then since angle $CPB+CBP=\alpha$, angle $CBP=\frac{1}{2}\alpha$ and $CP=CB$. Therefore ABC is the required triangle. Ed. E.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Suppose $ABCD$ the required trapezoid, AB being the shorter (upper) base, $BD=x$, one diagonal, $AC=BF=y$, BG=altitude$=h$. Produce DC to F so that $DF=DG+GF=DC+CF=a$, $x+y=c$, θ=given angle. $\sqrt{(x^2-h^2)}+\sqrt{(y^2-h^2)}=a$. $4x^2y^2=a^4+(x^2+y^2)^2+4a^2h^2-2a^2(x^2+y^2)$. But $x^2+y^2=c^2-2xy$. $\therefore 4(c^2-a^2)xy=4a^2h^2+(c^2-a^2)^2$. Now $xy\sin\theta=ah$.

$$\therefore 4a^2h^2-4ah\left(\frac{c^2-a^2}{\sin\theta}\right)=-(c^2-a^2)^2.$$

$$\therefore h=\frac{1}{2a}(c^2-a^2)\tan\tfrac{1}{2}\theta \text{ or } h=\tfrac{1}{2}(c^2-a^2)\cot\tfrac{1}{2}\theta.$$

$$\therefore xy=\frac{c^2-a^2}{4\cos^2\frac{1}{2}a} \text{ or } \frac{c^2-a^2}{\sin^2\frac{1}{2}\theta} \text{ and } x+y=c.$$

$$\therefore x=\tfrac{1}{2}c\pm\frac{\sqrt{(a^2-c^2\cos^2\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta} \text{ or } \tfrac{1}{2}c\pm\frac{\sqrt{(a^2-c^2\sin^2\frac{1}{2}\theta)}}{2\cos\frac{1}{2}\theta};$$

$$y=\tfrac{1}{2}c\mp\frac{\sqrt{(a^2-c^2\cos^2\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta} \text{ or } \tfrac{1}{2}c\mp\frac{\sqrt{(a^2-c^2\sin^2\frac{1}{2}\theta)}}{2\cos\frac{1}{2}\theta}.$$

Hence lay off $DF=a$, and draw AE parallel to DF at a distance h from it. With D as center and $DB=x$ or y draw DB; then $DF=y$ or x. Draw any line parallel to BF as AC, $A'C'$, $A''C''$, and join B, D to C, C', C'', A, A', A'', respectively. Then any one of the many trapezoids thus formed fulfill the required conditions, as is evident by drawing a figure.

IV. Solution by J. J. KEYES, Nashville, Tenn.

Construct the triangle EBF having BE=sum of diagonals, BF=sum of bases, and angle $E=\frac{1}{2}$ given angle. At F construct the angle EFD=angle E, FD meeting BE in D. Through D draw DM parallel to FB. Take any point C on BF, draw CA parallel to FD meeting DM in A. Join AB, DC. $ABCD$ is the required trapezoid. In proof, $AO=CF$. $\therefore AD+BC=BC+BD$=sum of bases. $AC=DF=DE$. $\therefore AC+BD=BD+DE$=sum of diagonals. The angle between BD and AC=angle BDF=2 angle E=given angle.

Also solved by Elmer Schuyler.

CALCULUS.

Problem number 181 was also solved by S. A. Corey and L. C. Walker.

183. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

Evaluate $\displaystyle\int_0^\infty \frac{\sin 2nx\,dx}{(a^2+x^2)\sin x}$.

Solution by G. B. M. ZERR, A. M., Ph. D.

Let $u=\int_0^\infty \frac{\sin 2nx\,dx}{(a^2+x^2)\sin x}$, then

$$\frac{du}{dn}=\int_0^\infty \frac{2x\cos nx\,dx}{(a^2-x^2)\sin x}, \quad \frac{d^2u}{dn^2}=-\int_0^\infty \frac{4x^2\sin 2nx\,dx}{(a^2+x^2)\sin x}=4a^2u-4\int_0^\infty \frac{\sin 2nx\,dx}{\sin x}.$$

$$\text{Let } v=8\int_0^\infty \frac{\sin nx\cos nx\,dx}{\sin x}=8\int_0^\infty \frac{e^{nx\sqrt{-1}}-e^{-nx\sqrt{-1}}}{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}\cos nx\,dx.$$

Let $x\sqrt{(-1)}=\pi y$, $dx=-\pi\sqrt{(-1)}dy$;

$$\therefore v=-8\pi\sqrt{(-1)}\int_0^\infty \frac{e^{\pi ny}-e^{-\pi ny}}{e^{\pi y}-e^{-\pi y}}\cos(\pi ny\sqrt{-1})dy$$

$$=-8\pi\sqrt{(-1)}\left(\frac{\sin \pi n}{e^{\pi n\sqrt{-1}}+2\cos \pi n+e^{-\pi n\sqrt{-1}}}\right)=-2\pi\sqrt{(-1)}\tan \pi n.$$

$\therefore v=0$, since $n=$any integer. $\therefore \frac{d^2u}{dn^2}-4a^2u=0.$

$u=Ae^{2an}+Be^{-2an}$. When $n=0$, $u=0$, and $B=-A$. $\therefore u=A(e^{2an}-e^{-2an})$.

When $n=1$, $u=\frac{\pi e^{-a}}{a}$ and $A=\frac{\pi e^{-a}}{a(e^{2a}-e^{-2a})}$.

$$\therefore u=\frac{\pi e^{-a}}{a}\cdot\frac{e^{2an}-e^{-2an}}{e^{2a}-e^{-2a}}=\frac{\pi e^{-a}}{a}\cdot\frac{\sinh 2an}{\sinh 2a}.$$

184. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

If $u=f(x, y)$; $\xi=e^x y$; $y\eta=e^x$; show that

$$\frac{d^2u}{dx^2}-y^2\frac{d^2u}{dy^2}-y\frac{du}{dy}=4\xi\eta\frac{d^2u}{d\xi\,d\eta}.$$

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

We have $x=\frac{1}{2}\log\xi+\frac{1}{2}\log\eta$, $y=\sqrt{(\xi/\eta)}$, and therefore

$$\frac{\partial}{\partial\xi}=\frac{\partial x}{\partial\xi}\cdot\frac{\partial}{\partial x}+\frac{\partial y}{d\xi}\cdot\frac{\partial}{\partial y}=\frac{1}{2}\left(\frac{1}{e^x y}\frac{\partial}{\partial x}+\frac{1}{e^x}\frac{\partial}{\partial y}\right), \text{ whence } 2\xi\frac{\partial}{\partial\xi}=\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

In a similar way we get

$$2\eta\frac{\partial}{\partial\eta}=\frac{\partial}{\partial x}=y\frac{\partial}{\partial y}.$$

$$\therefore 4\xi\eta\frac{\partial^2 u}{\partial\xi\,\partial\eta}=\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)u=\frac{\partial^2 u}{\partial x^2}-y\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial x\,\partial y}$$

$$-y\frac{\partial u}{\partial y}-y^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}-y\frac{\partial u}{\partial y}.$$

Also solved by G. B. M. Zerr.

MECHANICS.

170. Proposed by ELISHA S. LOOMIS, Ph. D., Berea, Ohio.

Two angles of iron, A_1CD and A_1CA_3, move freely on a pivot at C. Rods B_1A_1 and B_1A_3 are attached, respectively, at A_1 and at some point A_3 so that when B_1 moves along the rod CR, which is perpendicular to A_1A_4, CD and CA_3 shall coincide in position with CE which is perpendicular to rod KR. When angle A_1CD is 135° find CA_3 in terms of CA_1. Also find the following:

a) That value of CB_1 which will require least effort exerted at B_1 to cause CA_3 to take the position CA_4.

b) That value of CB_1 which will cause B_2A_2, if produced, to pass through the point A_1.

c) As CB_1 varies in value, what is the locus of the intersection of A_1B_1 and A_2B_2? Of B_1A_3 and B_2A_4?

d) Suppose angle A_1CD to be any other angle than 135°, then find CA_3 in terms of CA_1.

(For figure see pp. 177 and 212.)

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $CB_1=a$, $CA_1=r$, $CA_3=x$, $\angle A_1CA_2=\angle A_3CA_4=\beta$.

$\therefore A_1B_1=\sqrt{(a^2+r^2)}$.

d) $A_2B_2=A_1B_1=\sqrt{(a^2+r^2)}=\sqrt{(r^2+CB_2^2-2r.CB_2\sin\beta)}$.

$\therefore CB_2=r\sin\beta\pm\sqrt{(a^2+r^2\sin^2\beta)}$.

$A_4B_2=x^2+2r^2\sin^2\beta+a^2+2r\sin\beta\sqrt{(a^2+r^2\sin^2\beta)}$.

$A_3B_1=A_4B_2=x^2+a^2+2ax\sin\beta$.

$$\therefore x=\frac{r^2\sin\beta+r(a^2+r^2\sin^2\beta)}{a}.$$

When $\beta=\frac{1}{4}\pi$, $x=\dfrac{r\sqrt{2}\ [r+\sqrt{(2a^2+r^2)}]}{2a}$.

a) Let P=pull along A_3B_1, Q=resolved part of pull along A_3C.

Then $Q=P\cos CA_3B_1=P\cos\theta$

$$=\frac{P[2x+\sqrt{(2)}a}{2\sqrt{[x^2+a^2+ax\sqrt{2}]}}=\frac{Pr[r+a\sqrt{2}+\sqrt{(2a^2+r^2)}]}{[r+\sqrt{(2a^2+r^2)}]\sqrt{(a^2+r^2)}}$$

$$=\frac{Pr}{\sqrt{(a^2+r^2)}}+\frac{Pra\sqrt{2}}{[r+\sqrt{(2a^2+r^2)}]\sqrt{(a^2+r^2)}}=a\text{ minimum.}$$

Differentiating and reducing,

$$(r^3\sqrt{2}-2ar^2-2a^3)\sqrt{(2a^2+r^2)}+r^4\sqrt{2}+2a^4\sqrt{2}-4a^3r-2ar^3=0.$$

Let $r^3/a^3=y^3$.

$$\therefore (y^3\sqrt{2}-2ay^2-2)\sqrt{(2+y^2)}+y^4 2\sqrt{2}-4y-2y^3=0.$$

$\therefore y=0$ or $a=\infty$ for the minimum effort.

b) $CB_2=\dfrac{r+\sqrt{(2a^2+r^2)}}{\sqrt{2}}$, $A_2B_2=\sqrt{(a^2+r^2)}$, $A_1A_2=r\sqrt{(2-\sqrt{2})}$.

$$A_1B_2=\sqrt{[a^2+2r^2+r\sqrt{(2a^2+r^2)}]}=A_2B_2+A_1A_2=\sqrt{(a^2+r^2)}+r\sqrt{(2-\sqrt{2})}.$$

Let $a^2+r^2=y^2$.

$$\therefore \sqrt{[y^2+r^2+r\sqrt{(2y^2-r^2)}]}=y+r\sqrt{(2-\sqrt{2})}$$

or $r(1-\sqrt{2})+2y\sqrt{(2-\sqrt{2})}=\sqrt{(2y^2-r^2)}$.

$$\therefore y=\frac{r\sqrt{(2-\sqrt{2})}}{\sqrt{(2)}-1},\quad a=\frac{r}{\sqrt{[\sqrt{(2)}-1]}}.$$

c) $x/a-y/r=1$ is the equation to A_1B_1 .....(1).

$rx+y\sqrt{(2a^2+r^2)}-r^2\sqrt{2}-r\sqrt{2}\sqrt{(2a^2+r^2)}=0$ is equation to A_2B_2 .....(2).

From (1), $a=\dfrac{rx}{r+y}$; this in (2) gives

$$(y-r\sqrt{2})^2[2x^2+(r+y)^2]=(r+y)^2(r\sqrt{2}-x)^2 \text{ for first locus.}$$

$$\frac{2ax}{r\sqrt{2}[r+\sqrt{(2a^2+r^2)}]}+\frac{y\sqrt{2}}{r+\sqrt{(2a^2+r^2)}}=1 \text{ is equation to } A_4B_2 \text{ .....(3).}$$

$$\therefore 2ax+2ry=r\sqrt{2}[r+\sqrt{(2a^2+r^2)}] \text{ .....(4).}$$

$$[r^2-2a^2+r\sqrt{(2a^2+r^2)}]y=r[r+\sqrt{(2a^2+r^2)}]x-ar[r+\sqrt{(2a^2+r^2)}]$$

is equation to A_3B_1.....(5).

From (4),

$$a=\frac{r^2x\sqrt{2}-2rxy\pm r\sqrt{[4x^2y^2+2r^2x^2-2\sqrt{(2)}r^3y]}}{2x^2-r^2}.$$

Substituted in (5) this gives the second locus.

171. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Prove that the electrical capacity of an oblate ellipsoid of revolution is $\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$, where a and b are the equatorial and polar semi-diameters.

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Suppose the thickness of the shell to be $m\phi$, then the force due to the shell at any point P, external, is $4\pi\rho m\phi$. The force due to the shell is obtained from $-\dfrac{dn}{du}=\dfrac{4\pi\rho m\phi abc}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$, where dn is an element of the normal at ρ.

Taking dx, dy, and dz in the direction of the normal we get from the differentiation of $\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1$, $dl=2\phi dn$.

Hence $dv=\frac{2\pi\rho mabcdl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$, from which we get

$$v=2\pi\rho mabc\int_l^\infty \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}.$$

Replacing $4\pi\rho mabc$, the mass of the shell, by Q in the electrical case

$$v=\frac{Q}{2}\int_l^\infty \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}$$

$$\text{and } \frac{Q}{v}=\left[\tfrac{1}{2}\int_0^\infty \frac{dl}{\sqrt{[(a^2+l)(b^2+l)(c^2+l)]}}\right]^{-1}.$$

Since $l=0$ and $a=c>b$, we get, upon integrating,

$$\frac{Q}{v}=\sqrt{(a^2-b^2)}/\left(\tfrac{1}{2}\pi-\tan^{-1}\frac{b}{\sqrt{(a^2-b^2)}}\right)=\sqrt{(a^2-b^2)}/\left(\tfrac{1}{2}\pi-\sin^{-1}\frac{b}{a}\right)$$

$$=\sqrt{(a^2-b^2)}/\cos^{-1}(b/a).$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

The electrification at any point is proportional to the length of the perpendicular from the center on the tangent plane at that point. Let us denote it by mp. We must have $\int mpdS$ equal to the charge Q.

But $\int pdS=4\pi a^2 b$. $\therefore m=Q/4\pi a^2 b$.

The potential V at any point in the interior of the conductor is constant. At the center we have

$$V=\int\frac{1}{r}\frac{Qp}{4\pi a^2 b}2\pi r\sin\theta ds=\frac{Q}{2a^2 b}\int p\sin\theta ds=\frac{Q}{2a^2 b}\int_0^\pi r^2\sin\theta d\theta$$

$$=\tfrac{1}{2}Qb\int_0^\pi \frac{\sin\theta d\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{Q}{2\sqrt{(a^2-b^2)}}\tan^{-1}\left[\frac{\sqrt{(a^2-b^2)}}{b}\cos\theta\right]_\pi^0$$

$$=\frac{Q}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\frac{\sqrt{(a^2-b^2)}}{b}\right]=\frac{Q}{\sqrt{(a^2-b^2)}}\cos^{-1}(b/a).$$

$\therefore$ The capacity $Q/V=\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$.

AVERAGE AND PROBABILITY.

154. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Five points are taken at random in space. The chance that the fifth point may be included within the tetrahedron of which the other four points are the vertices is $\frac{1}{27}$.

Solution by the PROPOSER.

With the five points taken at random as vertices, five tetrahedrons may be formed.

Let A, B, C, D be the four points which form the greatest tetrahedron, and let E be the fifth point. Through the vertices A, B, C, D, pass the planes $B'C'D'$, $A'C'D'$, $A'B'D'$, $A'B'C'$, parallel, respectively, to the planes BCD, ACD, ABD, ABC. Now E must fall within the tetrahedron $D-ABC$; for if E fell without it, the tetrahedron $D-ABC$ would not be the greatest tetrahedron which could be formed with the five random points as vertices.

Consequently the volume of the tetrahedron $D'-A'B'C'$ represents the extent of all possible cases; that is, of all cases consistent with the assumption that $D-ABC$ is the greatest tetrahedron that could be formed with the five random points as vertices. The volume of the tetrahedron $D-ABC$ represents the total number of favorable cases. Hence the required chance is

$$\Delta = \frac{\text{the volume of tetrahedron } D-ABC}{\text{the volume of tetrahedron } D'-A'B'C'} = \frac{1}{27}.$$

156. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the average area of a triangle, the sum of whose sides is constant and equal to $2a$.

Solution by the PROPOSER.

Let x, y, $2a-x-y$ be the sides.

Then the area is $A=\sqrt{[a(a-x)(a-y)(x+y-a)]}$.

The limits of x are 0 and a; of y, $a-x$ and a. Putting Δ for the average area required, we have

$$\Delta = \int_0^a \int_{a-x}^a A\,dx\,dy \Big/ \int_0^a \int_{a-x}^a dx\,dy = \frac{2}{a^2}\int_0^a \int_{a-x}^a A\,dx\,dy.$$

Let $a-y=u$, $du=-dy$; the limits of u are x and 0.

$$\therefore \Delta = \frac{2}{a^2}\int_0^a \int_0^x \sqrt{[a(a-x)(xu-u^2)]}\,dx\,du = \frac{\pi}{4a^2}\int_0^a x^2\sqrt{[a(a-x)]}\,dx = \frac{4}{105}\pi a^2.$$

Solved with the same result by S. A. Corey, and G. B. M. Zerr.

DIOPHANTINE ANALYSIS.

122. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

If p is a prime $(p^4-1)(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

No satisfactory solution has been received.

MISCELLANEOUS.

145. Proposed by H. F. MacNEISH, Chicago, Ill.

Two complete 5-plane configurations in space having the same vertices are identical; in general two complete $(n+2)$-faces in n-space having the same vertices are identical.

Solution by the PROPOSER.

We proceed at once to the general case.

Definition. An $(n+2)$-face in n space is defined as the n-space configuration formed by $(n+2)$ $(n-1)$-spaces and the intersections with the restriction that no $i+2$ $(i+j)$-spaces have a common j-space.

Suppose one $(n+2)$-face has the following notation: The $(n-1)$-spaces are specified $A^{n-1}_{i_r}$ $(r=1 \ldots\ldots n+2)$, and any j-space has notation of the type $A^j{}_{i_1 \ldots\ldots i_{n-j}}$ and the incidence relations are fully specified by stating that $A^j{}_{i_1 \ldots\ldots i_{k-j}}$ lies in precisely every element of higher dimensions whose subscripts are all of the set $i_1 \ldots\ldots i_{k-j}$.

(I) *Any three collinear points are of the type*

$$A'_{i_1 i_2 \ldots\ldots i_{n-1}} = A^0{}_{i_1 i_2 \ldots\ldots i_n}\, A^0{}_{i_1 i_2 \ldots\ldots i_{n-1} i_{n+1}}\, A^0{}_{i_1 i_2 \ldots\ldots i_{n-1} i_{n+2}}.$$

We consider any two points with every possible third point and show that this type is the only possible type of three collinear points.

(a) $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1} i_{n+1}}$ $(n-1$ subscripts common). With these two points $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1} i_{n+2}}$ is collinear from the $(n+2)$-face $A^{n-1}_{i_r}$; any other third point will be of the type $A^0{}_{i_2 \ldots\ldots i_{n+1}}$, which will then be on the line $A'_{i_1 i_2 \ldots\ldots i_{n-1}}$ (being collinear with $A^0{}_{i_1 i_2 \ldots\ldots i_n}$ and $A^0{}_{i_1 i_2 \ldots\ldots i_{n-1} i_{n+1}}$ in $(n+2)$-face $A^{n-1}_{i_r}$) and then $n+1$ $(n-1)$-space $A^{n-1}_{i_1}$, $A^{n-1}_{i_2} \ldots\ldots A^{n-1}_{i_{n+1}}$ will pass through $A^0{}_{i_2 \ldots\ldots i_{n+2}}$ which contradicts the definition of an $(n+2)$-face.

(b) $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_2 i_4 \ldots\ldots i_{n+2}}$ $(n-2$ subscripts common) with $A^0{}_{i_1 i_2 \ldots\ldots i_{n-2} i_{n+1} i_{n+2}}$.

Now the line determined by $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_2 i_4 \ldots\ldots i_{n+2}}$ lies in the plane $A^2{}_{i_2 \ldots\ldots i_n}$, and the line determined by $A^0{}_{i_1 i_2 \ldots\ldots i_n}$, $A^0{}_{i_1 i_2 \ldots\ldots i_{n-2} i_{n+1} i_{n+1}}$ lies in the

plane $A^2_{i_2 \ldots i_{n-1}}$, and the line determined by $A^0_{i_2 i_4 \ldots i_{n+2}}$, $A^0_{i_1 i_3 \ldots i_{n+2} i_{n+1} i_{n+2}}$ lies in the plane $A^2_{i_2 \ldots i_{n-2} i_{n+1} i_{n+2}}$.

Therefore if $A^0_{i_1 i_2 \ldots i_n}$, $A^0_{i_2 i_4 \ldots i_{n+2}}$, $A^0_{i_1 i_3 \ldots i_{n-2} i_{n+1} i_{n+2}}$ are collinear, these three planes are coaxial, which contradicts the definition.

(b_2) $A^0_{i_1 i_2 \ldots i_n}$ $A^0_{i_2 i_4 \ldots i_{n+2}}$ with $A^0_{i_2 \ldots i_{n+1}}$. On this line will lie the point $A^0_{i_2 \ldots i_n i_{n+2}}$ for it is collinear with $A^0_{i_1 i_2 \ldots i_n}$ and $A^0_{i_2 \ldots i_{n+1}}$ in the $(n+2)$-face $A^{n-1}_{i_r}$ and at $A^0_{i_2 \ldots i_{n+2}}$ we then have $n+1$ (-1)-spaces concurring, which contradicts the definition.

Therefore the points are collinear only according to type (a); there are exactly ${}_{n+2}C_{n-1}$ such lines, and these are the lines of the $(n+2)$-face $A^{n-1}_{i_r}$.

Therefore if the xertices of two $(n+2)$-faces are identical the lines are identical.

....

....

....

(II) *Any $k+3$ k-spaces in the same $(k+1)$-spaces are of the type for $k=0$ ..... $n+2$.*

(a) $$A^{k+1}_{i_1 \ldots i_{n-k-1}} = \left\{ \begin{array}{l} A^k_{i_1 \ldots i_{n-k-1} i_{n-k}} \\ A^k_{i_1 \ldots i_{n-k-1} i_{n-k+1}} \\ A^k_{i_1 \ldots i_{n-k-1} i_{n-k+2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A^k_{i_1 \ldots i_{n-k-1} i_{n+2}} \end{array} \right.$$

For if only $n-k-2$ or less subscripts are common for any two k-spaces in the same $(k+1)$-space we would have k-spaces of the types $A^k_{i_1 \ldots i_{n-k-2} i_{i_m}}$, $A^k_{i_1 \ldots i_{n-k-2} i_{i_r}}$ lying in the same $(k+1)$-space. They intersect in a $k-1$)-space which will lie in $(n-k+2)$ $(n-1)$-spaces; *i. e.*, $A^{n-1}_{i_1}$, $A^{n-1}_{i_2}$, ..., $A^{n-1}_{i_{n-k-2}}$, $A^{n-1}_{i_j}$, $A^{n-1}_{i_m}$, $A^{n-1}_{i_r}$, $A^{n-1}_{i_s}$, which contradicts the definition that no $i+2$ $(i+k)$-spaces have a common k-space.

Therefore if the k-spaces of two $(n+2)$-faces are identical the $(k+1)$-spaces are identical $k=0$, ..., $n+2$.

Therefore the two $(n+2)$-faces are identical throughout.

PROBLEMS FOR SOLUTION.

ALGEBRA.

215. Proposed by EDWIN L. RICH, Lehigh University.

Solve (1).....$x/a+y/b+c/z=3$,
(2).....$x/a+b/y+z/c=3$,
(3).....$a/x+y/b+z/c=3$.

216. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Express by radicals the roots of $x^6+ax^4+bx^3+\frac{1}{4}a^2x^2+\frac{1}{2}abx+c=0$.

217. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the condition that $E\equiv x^5-bx^3+cx^2+dx-e$ shall be the product of a complete square and a complete cube.

218. Proposed by SAUL EPSTEEN.

Prove that $\sum_{r=0}^{n}\frac{c_r}{r+1}=\frac{2^{n+1}-1}{n+1}$, where c_r is the coefficient of x^r in the expansion of $(1+x)^n$.

GEOMETRY.

236. *Errata.* This problem should read: If two sides of a triangle pass through fixed points, the third side touches a fixed circle.

243. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

What is the equation to the curve on which lie the centers of the inscribed circles in the right-angled triangles of hypotenuse h?

244. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Upon the sides of a triangle as bases isosceles triangles with base angles of 30° are constructed. Show that the lines joining the vertices of these isosceles triangles form an equilateral triangle.

245. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

PCP', DCD' are conjugate diameters of an ellipse; PN, DM are the ordinates to the major axis at P and D; show $CM/PN=CN/DM=AC/BC$, and that AP and BD' are parallel, and that AP' is parallel to BD.

CALCULUS.

187. Proposed by L. T. JACKSON, St. Louis, Mo.

Find the area of the ellipse

$$x=a_1+a_2\cos\theta+a_3\sin\theta,$$
$$y=b_1+b_2\cos\theta=b_3\sin\theta.$$

188. Proposed by SAUL EPSTEEN.

Evaluate $\int \frac{y(y^2+2)dy}{(y^4+3y^2+3)\sqrt{(y^2+1)}}$.

AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Santa Barbara, Cal.

Three points are taken at random in a given circle, and a circle passed through them. The probability that the circle through the random points will be wholly in the given circle is $\frac{9}{35}$.

NOTES.

A number of good problems in Algebra, for solution, are desired.

Mr. T. H. Barton has been appointed instructor in mathematics in Dartmouth College.

Dr. E. L. Dodd has been appointed instructor in mathematics in the University of Iowa.

At Rutgers College Prof. E. A. Bowser has been made emeritus professor of mathematics.

Mr. A. Hall has been appointed instructor in mathematics in the United States Naval Academy.

Mr. H. V. Gummere has been appointed professor of mathematics in the Drexel Institute, Philadelphia.

Dr. H. C. Converse has been appointed instructor in mathematics in the Baltimore Polytechnic Institute.

Mr. E. D. Grant has been promoted to an assistant professorship in mathematics and physics at the Michigan College of Mines.

Mr. T. E. Gravatt and Mr. C. F. Sharp have been appointed to instructorships in mathematics in the Pennsylvania State College.

Dr. B. L. Newkirk has been appointed instructor, and Mr. A. J. Champreux assistant in mathematics at the University of California.

The following new members have recently been admitted to the American Mathematical Society: R. P. Baker, W. H. Bussey, H. A. Converse, A. M. Curtiss, G. R. Dean, E. L. Dodd, R. R. Fleet, E. D. Grant, J. E. Higden, L. E. Karpinski, O. C. Lester, Tultio Seir-Cevita, J. C. Symer, W. F. Monerieff, A. Ranum, C. S. Sisam, Adelaide Smith, Clara E. Smith, C. M. Snelling, Edward Study, D. T. Wilson.

The Index for Vol. XI will be sent out with the January number of Vol. XII.

BOOKS.

Exercises in Algebra. By Edward R. Robbins and Frederick H. Somerville, William Penn Charter School, Philadelphia, Penn. 8vo. Cloth Sides and Leather Back. 173 pages. New York and Chicago: The American Book Co.

This little book abounds in well selected exercises suitable for beginners in Algebra. F.

Practical Measurements in Magnetism and Electricity. By George A. Hoadley, A. M., C. E., Professor of Physics in Swarthmore College; Author of "A Brief Course in Physics." 8vo. Cloth, 111 pages. New York and Chicago: The American Book Co.

This book contains a fine collection of experiments in Magnetism and Electricity well suited to the requirements of students in the High School or Academy. F.

An Introduction to the Modern Theory of Equations. By Florian Cajori, Ph. D., Professor of Mathematics, Colorado College. 18vo. Cloth, ix+239 pages. Price, $1.75. New York: The Macmillan Co.

In addition to the usual matter treated in works on the Theory of Equations, Dr. Cajori has, in this treatise, devoted 24 pages to Substitutions, 5 pages to the Lagrangian Resolvents, 26 pages to Galois' Theory of Algebraic Numbers, 24 pages to Normal Domains, and 12 pages to Reduction of the Galois Resolvent by Adjunction.

The treatment of the Theory of Equations as here presented is clear, eminently scientific, and thoroughly modern. It cannot fail to satisfy the demands for a first-class work on the subject it treats. F.

Physical Laboratory Manual for Use in Schools and Colleges. By H. N. Chute, M. S., Author of "Practical Physics," etc., and Teacher of Physics in the Ann Arbor High School. Revised Edition. 8vo. Cloth, 266 pages. Price, 80 cents. Boston and Chicago: D. C. Heath & Co.

The experiments in this book may all be performed in the average High School Physical Laboratory, and will give the student a thorough understanding of the physical principles involved in each. F.

ERRATA.

Page 198, Mechanics, for 121 read 171.

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[Prepared by Saul Epsteen.]

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Grace's *The Algebra of Invariants*, MacMahon's *Elementary Plane Geometry*, Colaw's *School Algebra*, Higgins's *Lessons in Physics*, Bowden's *Elements of the Theory of Numbers*, Wells' *Advanced Course in Algebra*, Miller's *Laboratory Physics*, Montgomery's *An Elementary American History*, Smith's *Primary Arithmetic*, Taylor's *Plane Trigonometry*, Maurer's *Technical Mechanics*, *The Jones Series of Readers*, Sabine's *A Student's Manual of a Laboratory Course in Physical Measurements*, Snyder and Palmer's *One Thousand Problems in Physics*, Davis's *Elementary Physical Geography*, 148—150.

Halsted's *Rational Geometry*, 178.

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Robbins and Summerville's *Exercises in Algebra*, Hoadley's *Practical Measurements in Magnetism and Electricity*, Cajori's *An Introduction to the Modern Theory of Equations*, and Chute's *Physical Laboratory Manual*, 242.

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